Gauge Group TQFT and Improved Perturbative Yang-Mills Theory

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ABSTRACT

We reinterpret the Faddeev-Popov gauge-fixing procedure of Yang-Mills theories as the definition of a topological quantum field theory for gauge group elements depending on a background connection. This has the advantage of relating topological gauge-fixing ambiguities to the global breaking of a supersymmetry. The global zero modes of the Faddeev-Popov ghosts are handled in the context of an equivariant cohomology without breaking translational invariance. The gauge-fixing involves constant fields which play the role of moduli and modify the behavior of Green functions at subasymptotic scales. At the one loop level physical implications from these power corrections are gauge invariant.

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1 Introduction

In this paper we interpret the Faddeev-Popov gauge fixing method of Yang-Mills theories as the construction of a Topological Quantum Field theory (TQFT) for the gauge group. The gauge group element, the Faddeev-Popov ghost and anti-ghost together with the Nakanishi-Lautrup Lagrange multiplier field form the BRST quartet counting for zero degrees of freedom. We first define a topological BRST invariant partition function in the gauge group depending on a background Yang-Mills connection and thereafter average over background connections with a gauge invariant weight. We thus obtain a partition function that does not dependent on arbitrary local redefinitions of gauge group elements.

The manipulations in this construction are formally the same as those of the Faddeev-Popov procedure. However, the requirement of defining a consistent TQFT for the gauge group elements immediately suggests that one should investigate the possible breaking of the supersymmetry. This will permit us to consider the Gribov question [1] from a new point of view. We shall see that there are obstructions to the definition of the TQFT due to the existence of constant zero modes for the Faddeev-Popov operator $\partial \cdot D^A$. These zero modes occur for all transverse Yang-Mills connections A. If the theory is defined on a finite space-time volume, the constant ghost modes are normalizable and cannot be ignored. Unless they are consistently gauge fixed, they in fact lead to a vanishing partition function in covariant gauges. We remove these zero modes by introducing constant ghosts and ghosts of ghosts in the context of an equivariant cohomology. In contrast to pointed gauges [2] (where the ghost fields are fixed at a particular space-time point), our procedure is covariant and avoids breaking the translation invariance of space-time. We will not treat gauge-field dependent zero modes of the Faddeev-Popov operator in this paper.

We can however show that the mean values of some observables of the gauge group TQFT depend on global topological properties of the background connection A and we construct a BRST-exact observable with ghost number zero whose expectation value does not vanish on certain manifolds. The BRST symmetry is found to be (globally) broken for finite manifolds with topological properties which also imply a (topological) Gribov ambiguity [1, 2, 3]. This is an explicit verification of Fujikawa's previous conjecture [4] that the BRST-symmetry could be broken as a consequence of the Gribov ambiguity.

There is apriori no reason why the infinite volume limit should not be a smooth one, and we therefore believe that the constant modes should also be gauge-fixed in this limit. It should be stressed that our equivariant construction neither destroys the usual perturbative scheme of Yang-Mills theory nor its renormalisability. The constant ghosts however generate nonlocal interactions which can lead to power corrections in the usual perturbative Green functions if certain (bosonic) ghosts have vacuum expectation values in the infinite volume limit. The power corrections only become important as one goes away from the asymptotic kinematical domain where the usual Faddeev-Popov prescription is justified. We compute these corrections for the transverse gluon propagator perturbatively to one loop in generalized covariant gauges. At this level, they are found to be gauge independent provided one suitably identifies expectation values of the constant

(bosonic) ghosts.

2 TQFT in the gauge group

Consider the Euclidean Yang-Mills partition function in Landau gauge obtained by the usual Faddeev-Popov construction

$$\begin{aligned} Z[j] &= \textstyle\int [dA] \mathrm{Det}\,(\partial\cdot D^A)\delta(\partial\cdot A)\,\exp\,[\int dx\,(\mathcal{L}_{cl}(A) + j\cdot A)] \\ &= \textstyle\int [dA][db][dc][d\bar{c}]\,\exp\,[\int dx\,(\mathcal{L}_{cl}(A) + \bar{c}\partial\cdot D^A c - b\partial\cdot A + j\cdot A)] \end{aligned} \tag{2.1}$$

Here A is the Yang-Mills connection and D^A the associated covariant derivative. $\mathcal{L}_{cl}$ is a gauge invariant Lagrange density, for instance the local curvature, $\mathcal{L}_{cl} = \frac{1}{4}\mathrm{Tr}\,(F^2_{\mu\nu}(A))$. In the last expression of (2.1) the delta function and nonlocal determinant in the measure are represented by functional integrals over the Lagrange multiplier $b(x)$ and the Grassmannian Faddeev-Popov ghost $c(x)$ and $\bar{c}(x)$.

The partition function (2.1) suffers from the Gribov ambiguity [1] and our aim is to define a TQFT that leads to an improved generating functional which is protected from the generic zero modes of the operator $\partial\cdot D$. The field variables of this TQFT are the gauge group elements $U(x)$, which in the $SU(n)$ case are unitary matrices depending on the space-time position x. The gauge transformation of a connection $A(x) = A_\mu(x)dx^\mu$, a one-form valued in the Lie-Algebra $\mathcal{G}$ of the gauge group G, is

$$A^U(x) = U^{-1}(x)A(x)U(x) + U^{-1}(x)dU(x) \tag{2.2}$$

A gauge theory is a quantum field theory which does not distinguish between representations of the field variables by the configuration $\{A(x)\}$ or $\{A^U(x)\}$. In the language of TQFT, it is a theory defined by a Lagrangian density depending locally on U in such a way that the expectation value of certain "observables" does not depend on arbitrary local redefinitions of the U's. Inspired by our present understanding of TQFT's [5], we thus tentatively consider the following topological BRST symmetry

$$\begin{aligned} sU(x) &= \Psi(x) \\ s\Psi(x) &= 0 \\ s\bar{\Psi}(x) &= b(x) \\ sb(x) &= 0 \end{aligned} \tag{2.3}$$

The fields $\Psi(x)$ and $\bar{\Psi}(x)$ are topological anticommuting ghosts and antighosts for $U(x)$, and $b(x)$ is a Lagrange multiplier field.

Since $U(x)$ is a group element, it has an inverse, and we can introduce the $\mathcal{G}$-valued ghost $c(x)$ by

$$\Psi(x) = U(x)c(x) \tag{2.4}$$

With this change of variables, one has

$$\begin{aligned}
sU(x) &= U(x)c(x) \\
sc(x) &= -\frac{1}{2}[c(x), c(x)] \\
s\bar{c}(x) &= b(x) \\
sb(x) &= 0
\end{aligned} \tag{2.5}$$

We have redefined $\bar{\Psi}$ into $\bar{c}$ for the sake of notational consistency. One obviously has $s^2 = 0$. For the particular case of unitary groups, which we will exclusively discuss in the following, the relation $UU^\dagger = 1$ becomes consistent with the s-operation provided $c^\dagger = -c$. The definition of the hermitian conjugate ghost $c^\dagger$ is thus independent of U. The change of variable expressed by (2.4) is also conceptually interesting, since it provides us with an intrinsic geometric definition of the Faddeev-Popov ghost, with

$$c(x) = U^{-1}(x)sU(x) \tag{2.6}$$

We introduce the gauge field configuration $A(x)$ as a background, that is

$$sA(x) = 0 \tag{2.7}$$

Using the definition (2.2), one finds

$$sA^U(x) = -D^{A^U}c(x) \tag{2.8}$$

where

$$D^A = d + [A(x), \ ] \tag{2.9}$$

is the covariant derivative.

Let us define the TQFT for the fields $\{U(x)\}$ by a BRST-exact local action of the form

$$S_A[U, c, \bar{c}, b] = \int dx\, sW(A^U, c, \bar{c}, b) \tag{2.10}$$

and the corresponding partition function

$$\mathcal{Z}[A] = \int [dU][dc][d\bar{c}][db] \ \ \exp S_A \tag{2.11}$$

W in (2.10) is a local functional of the fields $c, \bar{c}, b$ and A with total ghost number -1. By expanding sW as a function of all fields, one obtains an action which is s-invariant, since $s^2 = 0$. The resulting BRST symmetry can be interpreted as a twisted version of supersymmetry [5]. If this supersymmetry is unbroken, the usual arguments based on the definitions of bosonic and fermionic path integration would show that $\mathcal{Z}[A]$ is a sum of ratios of determinants, such that

$$\mathcal{Z}[A] \neq 0 \qquad \text{and} \qquad \frac{\delta \mathcal{Z}[A]}{\delta A} = 0 \tag{2.12}$$

Consider the following choice for the topological action,

$$
\begin{aligned}
S_A &= -2\int dx\, s\mathrm{Tr}\,\Big(\bar{c}(x)\partial\cdot A^U(x)\Big) \\
&= 2\int dx\, \mathrm{Tr}\,\Big(\bar{c}(x)\partial\cdot D^{A^U}c(x) - b\partial\cdot A^U\Big)
\end{aligned}
\tag{2.13}
$$

The expectation value of a gauge invariant field functional $\mathcal{O}[A]$ is defined as

$$
\langle\mathcal{O}[A]\rangle \propto \int [dA]\mathcal{O}[A]\ \exp\,[\int dx\,\mathcal{L}_{cl}]\ \int [dU][dc][d\bar{c}][db]\ \exp S_A
\tag{2.14}
$$

Suppose now that the BRST symmetry is unbroken. Then (2.12) is true and (2.14) factorizes into the expectation value of $\mathcal{O}(A)$ with respect to the gauge invariant measure $[dA]\ \exp\int dx\,\mathcal{L}_{cl}$ and a nonvanishing normalization. Gauge invariance of this measure and the observable $\mathcal{O}$ furthermore implies that the change of variables

$$
A \to A' = U^\dagger AU + U^\dagger dU
\tag{2.15}
$$

decouples the integration over the volume of the gauge-group and one has

$$
\langle\mathcal{O}[A]\rangle \propto \int [dA'][dc][d\bar{c}]\mathcal{O}[A']\ \exp\int dx\ \Big(\mathcal{L}_{cl}(A') + 2\mathrm{Tr}\,(\bar{c}\partial\cdot D^{A'}c - b\partial\cdot A')\Big)
\tag{2.16}
$$

We have thus reproduced the Faddeev-Popov construction by coupling a supposedly trivial TQFT in the gauge group to the gauge-invariant Yang-Mills theory. We have used the gauge-invariance of the measure $[dA]$, of the classical action $\int dx\,\mathcal{L}_{cl}$ and of the observable $\mathcal{O}$ to factorize the integration over the gauge-group.

It is clear that all defects of the Faddeev-Popov construction must be present in this derivation, which at first sight is only a change of terminology. Indeed, the definition (2.1) of $Z[j]$ only makes sense perturbatively since the condition $\partial\cdot A = 0$ does not select unambiguously the representative A of the gauge field [1, 2]. This is due to the existence of disconnected gauge transformations which imply a nontrivial moduli-space for the equation $\partial\cdot A = 0$. Thus, the statement (2.12) must be wrong in our derivation, and the point we wish to make is that the Gribov ambiguity is related to a breaking of the supersymmetry of the topological action (2.13).

The origin of the SUSY-breaking is quite clear: the operator $\partial\cdot D^A$ has zero modes. Thus the quadratic form $\int dx\,\bar{c}(\partial\cdot D^A)c$ is degenerate and there is a deficit in the supersymmetric compensations which would otherwise enforce (2.12). Having $\mathcal{Z}[A] = 0$ on a domain of the connection A with nonzero measure would destroy the meaning of the Faddeev-Popov construction.

We can distinguish two types of ghost and antighost zero modes, the constant ones and those which are space-time dependent. The nonconstant zero modes generally depend

quite strongly on the connection A. The latter type of zero modes can be avoided in the formulation of the TQFT by choosing an appropriate background A. They could however be important in the YM-theory, if the set of connections A with such zero modes has non-vanishing measure (see ref. [6]). Generally however, their existence does not seem to jeopardize the definition of the Yang Mills path integral itself [7, 8] although they could render a perturbative evaluation of it untractable [6].

We will focus on the constant zero modes, since they are the only ones which are an obstruction to the definition of the TQFT (2.11), for *any* background connection A. These zero modes are present for all transverse connections A (which certainly is a set of nonvanishing measure). Due to the fact that, $\partial \cdot D^A = D^A \cdot \partial$, for transverse connections, the YM-action in Landau gauge has the obvious on-shell symmetry,

$$\begin{aligned} c(x) &\to c(x) + const. \\ \bar{c}(x) &\to \bar{c}(x) + const. \end{aligned} \tag{2.17}$$

The associated zero modes are normalizable if the base-manifold is compact and imply that eq. (2.12) does not hold for any A in this case. The above symmetry (2.17) is a consequence of the rigid gauge invariance of the covariant gauge fixing and the partition function (2.11) of the TQFT with the action (2.13) is proportional to the Euler number $\chi(SU(n)) = 0$ of the global $SU(n)$ group manifold of fixed points [5]. This destroys the possibility to reach the Faddeev-Popov action, since the partition function of the gauge group TQFT vanishes. For finite space-time volume the symmetry (2.17) therefore has to be removed and we believe this is also necessary to define a smooth infinite volume limit.

The situation is analogous to that in string theory where one cannot globally gauge-fix the 2-dimensional metric to a background metric. The field theory signals this obstruction by the existence of zero modes for the reparametrization antighosts. The remedy is well-known. It consists in weakening the over gauge-fixing by introducing finite integrations over constant moduli, genus by genus [9]. This procedure can be interpreted as a BRST invariant gauge fixing term for the global zero modes of the Faddeev-Popov operator of string theory [10].

The analogy with string theory suggests that one should also gauge-fix in a BRST-invariant way the degeneracy of the Yang-Mills action with respect to constant translations of the ghosts and antighosts. The method we will use is covariant and fixes the global symmetry while preserving invariance under space-time translations (unlike pointed gauges [2, 11]). It is then possible to consistently define the partition function of the gauge-fixed Yang-Mills theory by averaging over the (background) connection of the gauge group TQFT in a gauge invariant way. This gauge-fixing therefore commutes with space-time symmetries which after all is the main reason for considering covariant gauges and their associated ghosts.

3 Gauge-fixing the global zero modes

To define the Yang-Mills theory, we must therefore include in the BRST algebra the symmetry (2.17). The theory will involve a system of $\mathcal{G}$-valued global ghosts associated with the constant modes of the Faddeev-Popov ghosts. The geometrically meaningful sector of the BRST symmetry has positive ghost number. To isolate consistently the gauge-fixing of the constant ghost zero modes from the usual Yang-Mills gauge-fixing, we consider the following equivariant BRST symmetry modulo constant gauge transformations

$$\begin{aligned} sU(x) &= U(x)(c(x)+\omega) \\ sc(x) &= -\frac{1}{2}[c(x),c(x)] - [\omega, c(x)] - \phi \\ s\omega &= -\frac{1}{2}[\omega,\omega] + \phi \\ s\phi &= -[\omega,\phi] \end{aligned} \tag{3.1}$$

where ω and ϕ are x-independent. The commuting $\mathcal{G}$-valued constant ghost of ghost ϕ corresponds to the anticommuting parameter of the translational symmetry for the ghost $c(x)$, $c(x) \to c(x) + constant$. Moreover the positive degrees of freedom carried by ϕ compensate the negative ones carried by the $\mathcal{G}$-valued anticommuting constant ghost ω and thus the effective number of degrees of freedom remains unchanged. In other words, one can intuitively justify introducing an equivariant cohomology to cure the redundancy in the replacement of $c(x)$ by $c(x)+\omega$.

The gauge conditions we impose are

$$\begin{aligned} \partial\cdot A^U(x) &= 0 \\ \int dx \;\; c(x) &= 0 \\ \int dx \;\; \bar{c}(x) &= 0 \end{aligned} \tag{3.2}$$

These choices preserve the translation and the Euclidean invariance of space-time.

The topological action is obtained by implementing the constraints (3.2) with Lagrange multipliers $b(x)$, $\bar{\sigma}$ and γ having ghost number 0, -1 and 1 respectively. One can extend the action of s to these multiplier fields without destroying the nilpotency by introducing two more $\mathcal{G}$-valued constant ghost fields $\bar{\gamma}$ and σ. The canonical dimensions and ghost numbers of the fields are summarized in Table 1

field	$A(x)$	$U(x)$	$c(x)$	$\bar{c}(x)$	$b(x)$	ϕ	ω	σ	$\bar{\sigma}$	$\bar{\gamma}$	γ
dim	1	0	0	2	2	0	0	4	4	2	2
$\phi\Pi$	0	0	1	-1	0	2	1	-2	-1	0	1

Table 1. Dimensions and ghost numbers of the fields.

and the action of the BRST-operator s on all fields is

$$\begin{aligned}
sA_\mu(x) &= 0 \\
sU(x) &= U(x)\omega + U(x)c(x) \\
sc(x) &= -[\omega, c(x)] - \frac{1}{2}[c(x), c(x)] - \phi \\
s\omega &= -\frac{1}{2}[\omega, \omega] + \phi \\
s\phi &= -[\omega, \phi] \\
s\bar{c}(x) &= -[\omega, \bar{c}(x)] + b(x) \\
sb(x) &= -[\omega, b(x)] + [\phi, \bar{c}(x)] \\
s\sigma &= -[\omega, \sigma] + \bar{\sigma} \\
s\bar{\sigma} &= -[\omega, \bar{\sigma}] + [\phi, \sigma] \\
s\bar{\gamma} &= -[\omega, \bar{\gamma}] + \gamma \\
s\gamma &= -[\omega, \gamma] + [\phi, \bar{\gamma}]
\end{aligned} \tag{3.3}$$

It is straightforward to show that this BRST-operator is nilpotent on any element of the graded algebra constructed from the fields of Table 1

$$s^2 = 0 \tag{3.4}$$

Notice that one has

$$sA^U_\mu(x) = D^{A^U}_\mu c(x) - [\omega, A^U_\mu(x)] \tag{3.5}$$

We thus see that ω generates constant gauge transformations for all other fields of the BRST algebra. Moreover, the introduction of ω permits us to handle the invariance of $\partial \cdot A^U = 0$ with respect to constant gauge transformations. The BRST-exact topological action which implements the constraints (3.2) is

$$\begin{aligned}
S_A = sW_A &= 2\int_{\mathcal{M}} dx\ s\mathrm{Tr}\,[\partial^\mu \bar{c}(x) A^U_\mu(x) + \bar{\gamma}\bar{c}(x) + \sigma c(x)] \\
&= 2\int_{\mathcal{M}} dx \mathrm{Tr}\ \ [\partial^\mu b(x) A^U_\mu(x) - \partial_\mu \bar{c}(x) D^{A^U}_\mu c(x) \\
&\qquad + \gamma\bar{c}(x) + \bar{\gamma} b(x) + \bar{\sigma} c(x) - \frac{1}{2}\sigma[c(x), c(x)] - \sigma\phi\ ]
\end{aligned} \tag{3.6}$$

The nilpotency of s guarantees that S_A is BRST invariant. Due to this invariance the constant modes of the fields $b(x)$ and $\bar{c}(x)$ are simultaneously eliminated. Note that for an *abelian* group the bosonic ghosts σ and ϕ decouple completely.

S_A is independent of the anti-commuting ghost ω, generating constant gauge transformations. As a consequence, the partition function

$$\mathcal{Z}[A] = \int [dU][dc][d\bar{c}][db] d\phi d\omega d\sigma d\bar{\sigma} d\bar{\gamma} d\gamma\ e^{S_A} \tag{3.7}$$

vanishes due to the Grassmann-integration over ω^a. Notice that the integration over the corresponding bosonic zero modes of constant $SU(n)$ transformations only gives a factor proportional to the finite volume of the (global) $SU(n)$ group manifold. Table 1 shows that the measure in (3.7) has net ghost-number n^2-1. For a nonzero partition function $\mathcal{Z}[A]$, one has to absorb the excess Grassmann modes by inserting a factor $\prod_{a=1}^{n^2-1}\omega^a$ in the measure. Equivalently, one can drop the $d\omega$ integration in (3.7).

The action S_A is invariant under the transformations s_ω, parametrized by ω

$$\begin{aligned} s_\omega \omega &= 0 \\ s_\omega &\equiv s \ \text{ on all fields except } \omega \end{aligned} \tag{3.8}$$

which here plays the role of an external (Grassmann-)parameter. s_ω in general is not a nilpotent operation on all fields, since for instance

$$s_\omega^2 c(x) = [\omega^2 - \phi, c(x)] \neq 0 \tag{3.9}$$

The elimination of the constant ghost ω from the partition function can also be achieved by a more conventional gauge-fixing, using the additional fields defined in Table 2,

field	α	$\bar{\omega}$	β
dim	0	4	4
$\phi\Pi$	0	-1	0

Table 2. Dimensions and ghost numbers of the additional fields.

with s extended by

$$\begin{aligned} s\alpha &= 0 \\ s\bar{\omega} &= \beta \\ s\beta &= 0 \end{aligned} \tag{3.10}$$

Including these fields in the formalism, one can add the following BRST-exact action to S_A

$$\begin{aligned} S_{GF} &= 2\int dx\, s\left(\bar{\omega}(\alpha + \ln U(x))\right) \\ &= 2\int dx\, \left(\beta(\alpha + \ln U(x)) - \bar{\omega}(\omega + c(x))\right) \end{aligned} \tag{3.11}$$

This action, when inserted in the path integral, allows us to eliminate the fields $\omega, \bar{\omega}, \beta, \alpha$ by Gaussian integrations which yield ratios of equal normalisation factors, i.e. a factor of one. One can thus formally justify dropping the integration over ω in (3.7) at the price of introducing a multivalued function of $U(x)$ in an intermediate step.

Whichever explanation one prefers for the absence of ω in the final action (3.6), one is forced to only consider expectation values of ω-independent BRST invariant functionals. These functionals are globally gauge invariant since ω generates constant gauge transformations. We therefore define observables as elements of the equivariant cohomology Σ,

$$\Sigma = \{\mathcal{O} : \frac{\partial \mathcal{O}}{\partial \omega^a} = 0; s\mathcal{O} = 0, \mathcal{O} \neq sF\} \tag{3.12}$$

where F is itself ω-independent. Notice that the action of s_ω on ω-independent and globally gauge-invariant functionals is equivalent to that of s. The definition of physical observables can be more restrictive. For instance one may require that such an observable also has vanishing ghost number.

4 Observables in the gauge group TQFT and global breaking of the BRST symmetry

According to the last section, the physically interesting expectation values are

$$\langle \mathcal{O} \rangle_A = \int [dU][dc][d\bar{c}][db] d\phi d\sigma d\bar{\sigma} d\bar{\gamma} d\gamma \; \mathcal{O} \; e^{S_A} \tag{4.1}$$

where $\mathcal{O}$ belongs to Σ. In general, $\langle \mathcal{O} \rangle_A$ is independent of local variations of the background connection A since one has

$$\frac{\delta}{\delta A} \langle \mathcal{O} \rangle_A = \left\langle \mathcal{O} \frac{\delta}{\delta A} S_A \right\rangle_A = \left\langle \mathcal{O} s \frac{\delta}{\delta A} W_A \right\rangle_A = -\left\langle (s\mathcal{O}) \frac{\delta}{\delta A} W_A \right\rangle_A = 0, \; \forall \mathcal{O} \in \Sigma \tag{4.2}$$

We will show in the following that the constant fields introduced to gauge-fix global zero modes of the Faddeev-Popov ghosts and eventually construct a well-defined Yang-Mills partition function, also imply the existence of nonlocal observables depending on global properties of A. The existence of such observables does not invalidate the local equation (4.2), but it does prevent us from extending (4.2) to the statement that the expectation value of an observable is globally independent of the background A.

The existence of TQFT observables follows from Chern identities which imply descent equations for elements of Σ. It is convenient to introduce the 1-form,

$$L = U^{-1} dU \tag{4.3}$$

which is a flat connection associated with the unitary field $U(x)$

$$dL + L^2 = 0 \tag{4.4}$$

$\mathrm{Tr}\, L^3$ is a globally gauge invariant d-closed but not d-exact 3-form with vanishing ghost

number. It is the starting point of the descent chain,

$$
\begin{array}{rclcrcl}
0 & = & -d\Omega_3^0 & \qquad & \Omega_3^0 & = & \mathrm{Tr}\, L^3/3 \\
s\Omega_3^0 & = & -d\Omega_2^1 & & \Omega_2^1 & = & \mathrm{Tr}\, cL^2 \\
s\Omega_2^1 & = & -d\Omega_1^2 & & \Omega_1^2 & = & \mathrm{Tr}\, (c^2-\phi)L \\
s\Omega_1^2 & = & -d\Omega_0^3 & & \Omega_0^3 & = & \mathrm{Tr}\, (c^3/3 - c\phi) \\
s\Omega_0^3 & = & \Omega_0^4 & & \Omega_0^4 & = & \mathrm{Tr}\, \phi^2 \\
s\Omega_0^4 & = & 0 & & & &
\end{array}
\tag{4.5}
$$

The lower and upper indices of a form Ω denote respectively its form degree and its ghost number. Note that consistency requires that $d\Omega_0^4 = 0$ which is obviously true since ϕ is a constant field. The integrals over p-cycles of each p-form ($1 \leq p \leq 3$) in these descent equations are observables since they are elements of the equivariant cohomology Σ.

We can therefore identify the winding number $\mathcal{O}_0[U]_{\Gamma_3}$ of the map $U : \Gamma_3 \to SU(n)$ as an observable with ghost number zero,

$$
\mathcal{O}_0[U]_{\Gamma_3} = \frac{1}{8\pi^2}\int_{\Gamma_3} \Omega_3^0 = \frac{1}{24\pi^2}\int_{\Gamma_3} \mathrm{Tr}\, L^3 \tag{4.6}
$$

In what follows we will only consider this observable. It reflects certain topological properties of the space-time manifold $\mathcal{M}$: it vanishes for manifolds which do not admit any 3-cycle $\Gamma_3 \subset \mathcal{M}$ with $\pi_3(\Gamma_3) = \mathbf{Z}$. The winding number can for instance be nonzero on space-time manifolds with the topology $S_3 \times S_1$. On the other hand, it will vanish for the hypertorus.

The other forms in (4.5) with lower degree encode topological information even for manifolds where the winding number is trivial. This encoding is however quite abstract since the corresponding observables have nonzero ghost number. One can moreover also construct observables with forms obtained by ascending from invariants of the constant ghost ϕ with higher ghost number,

$$
\Omega_0^{2n} = \mathrm{Tr}\, \phi^n \Rightarrow \Omega_0^{2n-1} = \mathrm{Tr}\, c\phi^{n-1} + \ldots \Rightarrow \ldots \tag{4.7}
$$

One of the more interesting forms obtained in this way is the 4-form with ghost number 1

$$
\Omega_4^1 = \mathrm{Tr}\, cL^4 \tag{4.8}
$$

which is related to the potential ABBJ anomaly and whose chain terminates with $\mathrm{Tr}\, \phi^3$.

Using the winding number (4.6), we will now show that there is a global dependence of the gauge group TQFT on the connection A for certain manifolds. Consider a space-time manifold $\mathcal{M}$, which has at least one three-dimensional cycle $\Gamma_3 \subset \mathcal{M}$ with $\pi_3(\Gamma_3) = \mathbf{Z}$ and on which a unitary field $g(x)$ is defined which maps $g : \Gamma_3 \to SU(n)$ with a given winding number $\mathcal{O}_0[g]_{\Gamma_3} = w \neq 0$. Using the fact that the topological action S_A of (3.6)

only depends on U through A^U, basic group properties and (4.1) imply

$$\begin{aligned}\langle \mathcal{O}_0[U]_{\Gamma_3}\rangle_{A^g} &= \left\langle \mathcal{O}_0[g^\dagger U]_{\Gamma_3}\right\rangle_A = \langle \mathcal{O}_0[U]_{\Gamma_3}\rangle_A - \langle \mathcal{O}_0[g]_{\Gamma_3}\rangle_A \\ &= \langle \mathcal{O}_0[U]_{\Gamma_3}\rangle_A - w\,\langle 1\rangle_A \end{aligned} \tag{4.9}$$

To derive this formula, one uses the property that the winding number of the composition of two maps is the sum of their winding numbers. One also assumes that the path integral measure in (4.1) is gauge invariant – which is plausible in the absence of an ABBJ anomaly[4]. It is worth emphasizing that the only aspect of the equivariant construction which entered the derivation of (4.9) is the fact that the winding number (4.6) is an observable of the TQFT.

Singer [2] has shown that Gribov copies [1] of a gauge connection have to occur for manifolds which admit maps with nonvanishing winding number. Equation (4.9) implies that either $\mathcal{Z}[A]$ in (2.12) depends globally on A or vanishes for such manifolds and we will see below that this amounts to a breaking of the BRST symmetry. Precisely the same topological obstruction is therefore the cause of a Gribov ambiguity as well as the breaking of the BRST-symmetry. The relation (4.9) therefore strongly supports Fujikawa's [4] original conjecture that the two phenomena could be related.

In Appendix A we present topological arguments for the special case of an $SU(2)$ gauge group that the partition function $\langle \mathbf{1}\rangle_A$ no longer vanishes in the vicinity of $A=0$ for the equivariant TQFT defined by (4.1) with action (3.6). We now show that equation (4.9) in this case has the natural interpretation that the equivariant BRST symmetry we are discussing is globally (or spontaneously) broken. In view of Singer's [2] result, this would be an interesting way of characterizing the Gribov phenomenon. Our argument is quite formal however, since we consider the following nonlocal functional

$$\mathcal{O}_B = (W_A - W_{A^g})\sum_{n=1}^{\infty}\frac{1}{n!}\left(S_{A^g}-S_A\right)^{n-1} \tag{4.10}$$

$\mathcal{O}_B$ is globally gauge invariant and independent of the ghost ω. One therefore has,

$$s\mathcal{O}_B = -\sum_{n=1}^{\infty}\frac{1}{n!}\left(S_{A^g}-S_A\right)^{n} = 1 - \exp\left[S_{A^g}-S_A\right] \tag{4.11}$$

due to the definition (3.6) and the nilpotency of s. One can thus cast (4.9) in the form

$$\langle s\left\{\mathcal{O}_0[U]_{\Gamma_3}\mathcal{O}_B\right\}\rangle_A = w\,\langle 1\rangle_A \tag{4.12}$$

Provided the map $g:\Gamma_3\to SU(n)$ with winding number w exists, we therefore have a (nonlocal) s-exact functional whose expectation value for general background connections A does not vanish. This is the conventional indication for the spontaneous breakdown of a symmetry.

[4]Note however that topological quantities such as the winding number make little sense in a lattice analog of this model.

The results for the construction of the Yang-Mills theory in Landau gauge can be summarized as follows. By gauge fixing the constant zero modes of the ghosts, one obtains a generally well defined *nonvanishing* gauge group TQFT partition function which depends on global properties of the Yang-Mills background connection. One can in principle now proceed to a definition of the gauge theory by averaging over the background connection of the gauge group TQFT with a gauge invariant weight. In the following, we will choose other gauges for the gauge group TQFT in order to conveniently perform explicit computations and to check the gauge independence to some extent.

We will see that the constant ghosts introduced by the equivariant BRST (3.3) can modify the Yang-Mills perturbation theory in an interesting way. When they condense, they lead to power corrections for physical correlation functions at large momenta.

5 General covariant gauges

Since we are interested in practical computations to investigate possible modifications of perturbation theory by the constant ghosts, we will use the freedom of choosing different BRST invariant topological actions to eliminate most of the Lagrange-multiplier constraints of (3.6) in favor of interactions. Although stable under renormalization, (3.6) is not the most general BRST-exact renormalizable action one can construct with this field content.

By examining Table 1, one sees that the only local BRST exact terms of dimension four, ghost number zero and independent of ω which have been omitted in (3.6) are

$$\alpha \; s\mathrm{Tr}\, \bar{c}(x)b(x) = \alpha \; \mathrm{Tr}\,(b^2(x) + \phi[\bar{c}(x), \bar{c}(x)]) \tag{5.1}$$

and

$$\beta \; s\mathrm{Tr}\,[\bar{c}(x), \bar{c}(x)]c(x) = \beta \; \mathrm{Tr}\,(2b(x)[\bar{c}(x), c(x)] - \frac{1}{2}[\bar{c}(x), \bar{c}(x)][c(x), c(x)] - \phi[\bar{c}(x), \bar{c}(x)]) \tag{5.2}$$

where α and β are dimensionless gauge parameters. Note that $\mathrm{Tr}\, b^2$ alone would *not* be exact with respect to the BRST symmetry of eq. (3.3) because of the constant ghosts. We now observe that in more general covariant gauges where the TQFT of Landau gauge described by (3.6) is extended by the BRST-exact terms (5.1) and (5.2), the constant ghost ϕ usually couples to $\int d^4x\, [\bar{c}(x), \bar{c}(x)]$.

For $\alpha \neq 0$ the constraint $\partial \cdot A = 0$ is softened and replaced by a Gaussian dependence on the longitudinal part of the connection (in the gauge-theory longitudinal gluons only propagate in gauges with $\alpha \neq 0$). A nonvanishing value of the gauge parameter β generally introduces a local quartic ghost interaction, which leads to a more complicated perturbation theory. At the special value $\beta = \alpha$ this quartic ghost interaction vanishes after gaussian elimination of the field $b(x)$ and the resulting action is again bilinear in the ghosts $c(x)$ and $\bar{c}(x)$. Interestingly the constant ϕ-ghost also decouples from $[\bar{c}(x), \bar{c}(x)]$

at this point in the parameter space and ϕ as well as σ can be eliminated from the action. Moreover the elimination of $b(x)$ changes the Faddeev-Popov term $\bar{c}(x)\partial\cdot D^A c(x)$ into $\bar{c}(x)D^A\cdot\partial c(x)$. Upon rescaling $\bar{c}(x)\to\bar{c}(x)/\alpha$, $\bar{\sigma}\to\bar{\sigma}/\alpha$, $\gamma\to\gamma/\alpha$, the relevant topological action in gauges with $\beta=\alpha$ is,

$$
\begin{aligned}
S_A^{(1)}(\alpha) \;=\; & \frac{2}{\alpha}\int_{\mathcal{M}} dx \mathrm{Tr}\left[\frac{1}{2}(\partial\cdot A^U(x))^2+\bar{c}(x)\, D^{A^U}\cdot\partial c(x)-\right.\\
& \left. -[\bar{c}(x),c(x)]\bar{\gamma}+\frac{1}{2}\bar{\gamma}^2+\gamma\bar{c}(x)+\bar{\sigma}c(x)\right]
\end{aligned}
\tag{5.3}
$$

The other class of gauges where dynamical ghosts effectively only enter the action quadratically is when $\beta=0$. In these gauges one has

$$
\begin{aligned}
S_A^{(2)}(\alpha) \;=\; & \frac{2}{\alpha}\int_{\mathcal{M}} dx \mathrm{Tr}\left[\frac{1}{2}(\partial\cdot A^U(x))^2+\bar{c}(x)\partial\cdot D^{A^U}c(x)-\right.\\
& \left. -\frac{1}{2}\sigma[c(x),c(x)]-\frac{1}{2}\phi[\bar{c}(x),\bar{c}(x)]-\sigma\phi+\gamma\bar{c}(x)+\bar{\sigma}c(x)\right]
\end{aligned}
\tag{5.4}
$$

where we have again rescaled the fields $\bar{c}\to\bar{c}/\alpha$, $\sigma\to\sigma/\alpha$, $\bar{\sigma}\to\bar{\sigma}/\alpha$ in order to exhibit clearly that the loop expansion is an expansion in the gauge parameter α.

The models described by (5.3) respectively (5.4) are all stable under renormalization, essentially because one of the ghost momenta factorizes in the quartic ghost vertex, rendering it superficially convergent. $\beta=\alpha$ and $\beta=0$ are therefore fixed lines in the parameter space which intersect at the Landau gauge, ($\alpha=\beta=0$).

The dependence of the effective actions (5.3) and (5.4) on the constant ghosts $\bar{\gamma}$, respectively ϕ and σ, can be eliminated in favor of nonlocal quartic ghost interactions with zero momentum transfer. This is reminiscent of Cooper-pairing, especially since the ghosts obey Fermi-statistics. The analogy is perhaps particularly striking for the models described by effective actions (5.3) if we consider half the ghostnumber as the analog of the z-component of "spin": the "Cooper"-pair in this case has total ghostnumber 0 (i.e. $s_z=0$) (it is anyhow a scalar under rotations, since the ghost-fields themselves are and they are coupled to zero angular momentum). Such a "Cooper"-pair is furthermore "charged" since it transforms according to the adjoint representation of the gauge group. In contrast to the abelian case however, two such "Cooper"-pairs can form a state of vanishing chromoelectric charge, and the global gauge symmetry need not be broken if these "Cooper"-pairs of ghosts condense. The physical interpretation of the breakdown of the BRST-symmetry without breaking the global $SU(n)$ symmetry of the model could thus be the condensation of such "Cooper"-pairs. The other class of actions (5.4) can be thought of as a "Fierzed" description of the same effect: in this case there are two types of charged pairs with ghost number ± 2 (i.e. corresponding to electrons coupled to $s_z=\pm 1$). If they condense symmetrically, a similar physical picture may result. We will not pursue this amusing analogy further, but will show that the physical content of the two classes of models described by (5.3) and (5.4) is remarkably similar.

6 Modification of ordinary perturbation theory

In the following we use the gauge group TQFT partition functions defined with $S_A^{(1)}$ or $S_A^{(2)}$ instead of S_A, and average (4.1) over background connections A with a gauge invariant weight as explained in section 2. The motivation for this procedure is that it is not altogether trivial to implement Landau gauge in the mode expansion on a *finite* manifold. It is thus desirable for practical calculations to replace these constraints by a Feynman-type gauge-fixing term $(\partial\cdot A)^2$. We can explore the gauge dependence with the gauge parameter α and Landau gauge should be the limit $\alpha \to 0$ in either class of gauges. After the change of variables, $A^U \to A$ the unitary fields U decouple as previously and the resulting action is invariant under the BRST symmetry based on the equation

$$sA = -D^A c(x) - [\omega, A] \tag{6.1}$$

The new feature is that the perturbative computations will be modified by the interactions of the dynamical fields with the remaining constant ghosts in $S_A^{(1)}$ respectively $S_A^{(2)}$.

We are mainly concerned with the infinite volume limit of the $SU(n)$ gauge theories described by the classical actions

$$S_G^{(1,2)} = \mathrm{Tr} \int_{\mathcal{M}} \frac{1}{4g^2} F_{\mu\nu} F_{\mu\nu} + S_A^{(1,2)}(\alpha)\Big|_{U=\mathbf{1}} \tag{6.2}$$

To regularize ultra-violet divergences in the perturbative expansion, we use dimensional regularisation and renormalize according to the minimal scheme. From now on, D (without indices) denotes the dimension of spacetime ($D \leq 4$).

The actions $S_G^{(1,2)}$ depend on the usual dynamical ghost fields $c(x)$ and $\bar{c}(x)$ as well as on constant ghosts. The corresponding Euclidean field theory is defined on a finite D-dimensional manifold, which for simplicity will be taken to be a symmetric torus of extension L. The integration over the Grassmann variables γ^a and $\bar{\sigma}^a$ removes the constant modes of $c(x)$ and $\bar{c}(x)$ in the mode expansion. A perturbative treatment of the couplings between the dynamical ghosts and the constant *bosonic* ghosts $\bar{\gamma}$ or σ and ϕ would lead to severe infrared divergences of the perturbation series (just like a perturbative treatment of mass-terms would). We therefore propose to treat these bosonic fields like moduli on which the generating functional depends. The correlation functions have to be integrated over this space of moduli with a certain weight which in principle can be computed order by order in a loop expansion for the dynamical fields. The exact weight is not perturbatively computable, but we will see that one only has to know (or assume) a few expectation values of the constant ghosts to effectively parametrize the integration over the moduli-space in the infinite volume limit.

The general structure of the moduli-space is simplest for the gauge-theory defined by the classical action $S_G^{(1)}$, since it only depends on one constant bosonic ghost, $\bar{\gamma}$. If we only consider globally $SU(n)$-invariant expectation values, one can perform a global $SU(n)$-transformation to diagonalize $\bar{\gamma}$. The moduli space in this case is therefore the direct

product of the $SU(n)$ group manifold and an $(n-1)$-dimensional manifold described by the eigenvalues of $\bar{\gamma}$.

The moduli-space of the model described by the classical action $S_G^{(2)}$ is of higher dimension and considerably more complicated since it is described by ϕ and σ. One can again factor the $SU(n)$-group, but is left with a n^2-1 dimensional manifold, which furthermore has one flat direction if ghost number is conserved.

Following the general argument that functionals in the equivariant cohomology Σ can be chosen not to depend on BRST doublets [12], it is clear that physically interesting observables with vanishing ghost-number do not depend on the dynamical ghosts (nor on ϕ, σ, or $\bar{\gamma}$). We can therefore integrate over the dynamical ghosts $c(x)$ and $\bar{c}(x)$ that only appear quadratically in the effective action (6.2) without loosing any physical information. This results in an effective action Γ which is a nonlocal functional of the gauge connection A and the constant bosonic ghost(s). Since we are interested in the effects from the constant ghosts in a perturbative evaluation of gluonic correlation functions, we expand Γ in orders of the gauge field $A(x)$ as follows

$$\Gamma[\varphi, A(x)] = V_{\mathcal{M}}\Gamma^0(\varphi) + \frac{1}{2}\int dx \int dy\, A_\mu(x)^a \Gamma^{2\,ab}_{\mu\nu}(x-y,\varphi) A^b_\nu(y) + \ldots \tag{6.3}$$

The generic variable φ in (6.3) stands for all constant bosonic ghosts which have not been eliminated. The A-independent term Γ^0 in (6.3) is just the 1-loop effective action for the constant bosonic ghost(s). Note that we have explicitely indicated that this term is proportional to the volume $V_{\mathcal{M}}$ of space-time. The effective polarization Γ^2 in (6.3) depends on the constant ghost(s) in a highly nontrivial manner and leads to a correction of Γ^0 at the 2-loop level, etc. Since we cannot solve the gluonic part of the theory, we are not able to calculate the true weight for the integration over the moduli-space of the φ's. We can however quite independently of perturbation theory assert that the A-independent part of Γ is proportional to the volume of space-time. In the infinite volume limit only the maximum of the exact effective action $\Gamma^0(\varphi)$ is therefore relevant. This fact allows us to effectively perform the integration over the constant ghosts at any given finite order of the loop expansion for the gauge field by evaluating expectation values at the maximum of the full effective action Γ^0.

We will apply this procedure to compute the effective gluon polarization Γ^2 at the one loop-level as a function of the expectation value of the relevant φ's. The latter are therefore important parameters in a perturbative evaluation of the theory and we will show that nontrivial expectation values of the bosonic ghosts give rise to asymptotic power corrections in the loop expansion of correlation functions. Moreover, using two different classes of gauges corresponding to the choices $S_A^{(1)}$ and $S_A^{(2)}$ for the gauge group TQFT, we can test the gauge independance of physical correlation functions.

Let us first compute $\Gamma^0(\bar{\gamma})$ of (6.3) to one loop for the theory described by the action $S_G^{(1)}$. We take space-time to be a symmetrical torus $L \times L \times \ldots \times L$ of D dimensions[5]. The

[5]Apart from the finite volume we essentially follow the procedure of ref. [13]. Note that Γ^0 is the

classical contribution to $\Gamma^0(\bar{\gamma})$ is the quadratic term $-\frac{1}{2}\bar{\gamma}^a\bar{\gamma}^a/\alpha$, while the α-independent next order in the loop expansion is found by evaluating the infinite sum of one ghost loop diagrams with no external A fields shown in Fig. 1.

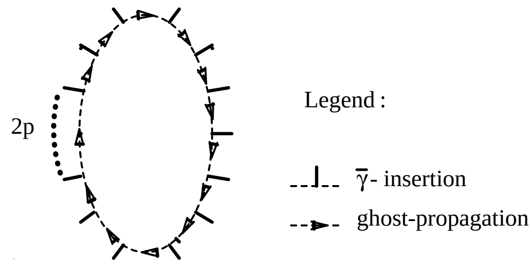

Fig.1: Nonvanishing 1-loop contribution to the effective moduli action: a ghost loop with $2p$ insertions of $\bar{\gamma}$-moduli.

Note that the antisymmetry of the structure constants f^{abc} of a nonabelian group implies that only loops with an even number of $\bar{\gamma}$-insertions give a nonvanishing contribution to the effective action. The loop with $2p$ insertions of the ghost $\bar{\gamma}$ (which we will call a p-loop) is proportional to

$$C_p = \mathrm{Tr}_{\,adj.}\hat{\bar{\gamma}}^{2p} \tag{6.4}$$

where we have used the notation $\hat{X}$ to denote the (antihermitian) matrix of the adjoint representation of X, $\hat{X}_{ab} = f^{abc}X^c$. For $SU(2)$ the traces (6.4) are simply powers of the single invariant $\bar{\gamma}^a\bar{\gamma}^a$

$$C_p(SU(2)) = 2(-\bar{\gamma}^a\bar{\gamma}^a)^p \tag{6.5}$$

The contribution from a single p-loop to the effective action is

$$F_D(p) = -\frac{1}{2p}\mathrm{Tr}_{\,adj.}\left(\frac{L^2\hat{\bar{\gamma}}}{(2\pi)^2}\right)^{2p} \sum_{\{n_1\dots n_D\}\neq\{0\dots 0\}} \left(n_1^2 + \dots + n_D^2\right)^{-2p} \tag{6.6}$$

The sum in (6.6) extends over all the sets of integers $\{n_1 \dots n_D\}$ describing the complete set of modes with momenta $k_\mu = 2\pi n_\mu/L$ of the dynamical ghosts. The contribution from the constant modes with $n_\mu = 0,\ \mu = 1,\dots,D$, to (6.6) is however eliminated by the integration over the ghosts $\bar{\sigma}$ and γ. $F_D(p)$ is therefore finite for $p > D/4$. The overall negative sign of (6.6) is due to the ghost statistics.

One can analytically continue to noninteger dimensions by casting $F_D(p)$ in integral form,

$$F_D(p) = -\mathrm{Tr}_{\,adj.}\left(\frac{L^2\hat{\bar{\gamma}}}{(2\pi)^2}\right)^{2p} \frac{1}{\Gamma(2p+1)}\int_0^\infty dx\, x^{2p-1}\left[(f(x))^D - 1\right] \tag{6.7}$$

where the function $f(x)$ for $x > 0$ is the convergent sum,

$$f(x) = \sum_{n=-\infty}^{\infty} e^{-xn^2} \tag{6.8}$$

negative of the effective potential of [13] and we are therefore interested in its maximum.

Although there is no analytic expression for $f(x)$ at arbitrary values of its argument, the reflection formula [14]

$$f(x) = \sqrt{\frac{\pi}{x}}\; f(\frac{\pi^2}{x}) \tag{6.9}$$

implies the asymptotic behaviour,

$$\begin{aligned} f(x\to\infty) &= 1+2e^{-x}+O(e^{-4x}) \\ f(x\to 0) &= \sqrt{\frac{\pi}{x}}(1+O(e^{-\frac{\pi^2}{x}})) \end{aligned} \tag{6.10}$$

Since the commuting ghosts $\bar{\gamma}^a$ can be treated as real numbers

$$v^2 = \frac{L^4\bar{\gamma}^a\bar{\gamma}^a}{(2\pi)^4} \tag{6.11}$$

is positive and the 1-loop contributions to the effective action can be summed over all values of $p \geq 1$. Adding this sum to the tree-level contribution gives

$$L^D\Gamma^0(\bar{\gamma}) = -\frac{(2\pi)^4 v^2 L^{D-4}}{2\alpha} - 4\int_0^\infty \frac{dx}{x}\sin^2(vx/2)\left[(f(x))^D - 1\right] \tag{6.12}$$

The asymptotic behaviour of $f(x)$ when $x \to 0$ determines the leading behaviour of the integral in (6.12) when $v \to \infty$. This is the large-volume limit we are interested in. One obtains in $D = 4-2\epsilon$ dimensions,

$$L^D\Gamma^0(\bar{\gamma}) \quad \stackrel{\mu L\to\infty}{\longrightarrow} \quad (\mu L)^D(-\tilde{v}^2)\left[\frac{(4\pi)^2\mu^{2\epsilon}}{2\alpha} + \frac{4}{4-2\epsilon}\tilde{v}^{-\epsilon}\cos(\frac{1}{2}\pi\epsilon)\Gamma(\epsilon-1)\right] \tag{6.13}$$

where we have introduced the *finite* scale μ since we are interested in the $\mu L \to \infty$ limit. The amplitude of $\bar{\gamma}^a\bar{\gamma}^a$ is now measured in terms of μ with

$$\tilde{v}^2 = \frac{\bar{\gamma}^a\bar{\gamma}^a}{(4\pi)^2\mu^4} \tag{6.14}$$

The term in square brackets of (6.13) does not depend on the large scale μL and we can expand it for $\epsilon \to 0$ (but an expansion of the factor $(L\mu)^D$ around $D = 4$ would make no sense, since we want to evaluate the effective action when $\epsilon\ln(L\mu) \gg 1$).

The $1/\epsilon$-term in the expansion of (6.13) can be removed by a counterterm of the form $(1-Z)V_{\mathcal{M}}\bar{\gamma}^a\bar{\gamma}^a/2\alpha$ in the classical action. To order $\hbar$ we thus have in the MS-scheme,

$$Z = 1 + \frac{2\hat{\alpha}\hat{g}^2}{(4\pi)^2\epsilon} \tag{6.15}$$

where we have expressed $\alpha = \hat{\alpha}\hat{g}^2\mu^{2\epsilon}$ in terms of the dimensionless gauge-parameter $\hat{\alpha}$ and coupling $\hat{g} = g\mu^{-\epsilon}$. The renormalization constants of the usual parameters and fields

of the gauge theories described by (6.2) are the same as those of the standard Yang-Mills theory (this will be verified shortly). The renormalization constant for the gauge parameter $\alpha = \hat{\alpha} g^2$ in the MS-scheme in the notation of [15] is

$$Z_\alpha = Z_3 Z_g^2 = 1 - \frac{3\hat{g}^2}{(4\pi)^2\epsilon} - \frac{\hat{\alpha}\hat{g}^2}{(4\pi)^2\epsilon} + O(\hbar^2) \tag{6.16}$$

for an $SU(2)$ gauge group[6]. Together with (6.15) Z_α determines the renormalization constant $Z_{\bar{\gamma}^2}$ between the bare and renormalized composites $(\bar{\gamma}^a\bar{\gamma}^a)_B$ and $(\bar{\gamma}^a\bar{\gamma}^a)_R$

$$(\bar{\gamma}^a\bar{\gamma}^a)_B = Z_{\bar{\gamma}^2}(\bar{\gamma}^a\bar{\gamma}^a)_R \tag{6.17}$$

One has

$$Z_{\bar{\gamma}^2} = Z_\alpha\, Z = 1 - \frac{3\hat{g}^2}{(4\pi)^2\epsilon} + \frac{\hat{g}^2\hat{\alpha}}{(4\pi)^2\epsilon} + O(\hbar^2) = \tilde{Z}_3^{-2} \tag{6.18}$$

where $\tilde{Z}_3$ is the renormalization constant for the kinetic term $\bar{c}\partial{\cdot}\partial c/\alpha$. This result reflects the fact at the one-loop level that the $\bar{c}\bar{\gamma}c/\alpha$-vertex in the action $S_G^{(1)}$ is not renormalized.

We may trade the dependence of the renormalized one-loop effective action on the dimensionless gauge parameter $\hat{\alpha}$ and scale μ for a new scale κ defined as follows

$$\ln\frac{\kappa^2}{4\pi\mu^2} = -\frac{(4\pi)^2}{2\hat{\alpha}\hat{g}^2} + 1 - \gamma_E \tag{6.19}$$

where γ_E is the Euler constant. The value $\bar{\gamma}^a\bar{\gamma}^a = \kappa^4$ maximizes Γ^0 at one loop. In terms of this scale, the one loop renormalized effective action Γ^0 for an $SU(2)$ gauge theory with classical action $S_G^{(1)}$ takes the simple form

$$L^4\Gamma^0(\bar{\gamma}^a\bar{\gamma}^a;\kappa^4) = -L^4\frac{\bar{\gamma}^a\bar{\gamma}^a}{32\pi^2}\ln\left[\frac{\bar{\gamma}^a\bar{\gamma}^a}{e\kappa^4}\right] \tag{6.20}$$

in four space-time dimensions.

The effective action (6.20) is real and has a unique maximum at

$$\bar{\gamma}^a\bar{\gamma}^a = \kappa^4 > 0 \tag{6.21}$$

We assume that the exact effective action will also have a nontrivial maximum. Since the effective action is proportional to the volume of space-time, the variance of $\bar{\gamma}^a\bar{\gamma}^a$ vanishes in the thermodynamic limit and it is sufficient to know the true expectation value $\langle\bar{\gamma}^a\bar{\gamma}^a\rangle$ to perturbatively evaluate any gauge boson correlation function in the equivariant cohomology of the $SU(2)$ theory. For a more general group than $SU(2)$, the knowledge of a larger number of expectation values of invariants of $\bar{\gamma}$ would be needed. This number equals the rank of the group. The one loop effective action calculated above indicates that the expectation value $\langle\bar{\gamma}^a\bar{\gamma}^a\rangle$ need not vanish and furthermore correctly predicts its

[6] Z_α at one loop is independent of the number N_F of quark flavours. This supports the idea that the gauge fixing of global zero modes only concerns the pure gauge boson sector

anomalous dimension. We will shortly examine the effect of a nontrivial expectation value $\langle\bar{\gamma}^a\bar{\gamma}^a\rangle$ on the transverse gluon polarization.

Let us first however address the issue of gauge-invariance by also calculating the one loop effective action for the constant ghosts σ and ϕ for the class of theories with classical actions of the type $S_G^{(2)}$. The one loop diagrams that have to be evaluated in this case are those of Fig. 2.

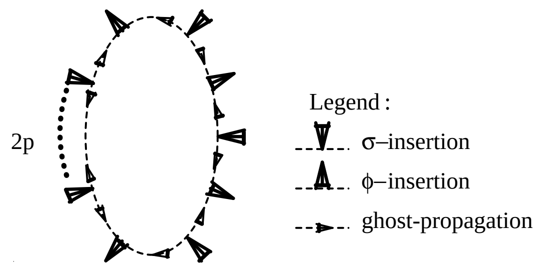

Fig.2: Nonvanishing 1-loop contribution to the effective moduli action: a ghost loop with p alternating insertions of ϕ and σ-moduli.

The structure of the interaction vertices in (5.4) requires that a nonvanishing loop has an equal number of ϕ and σ insertions, which furthermore alternate. A calculation similar to the previous one gives for the renormalized 1-loop effective action of the bosonic σ and ϕ ghosts,

$$L^4\Gamma^0(\phi^a\sigma^a;\bar{\kappa}^4) = L^4\frac{\sigma^a\phi^a}{32\pi^2}\ln\left[-\frac{\sigma^a\phi^a}{e\bar{\kappa}^4}\right] \tag{6.22}$$

where the gauge-group is again $SU(2)$ and the space-time volume is large.

The unique maximum of the effective action (6.22) occurs for

$$\sigma^a\phi^a = -\bar{\kappa}^4 < 0 \tag{6.23}$$

and the relation between the parameter $\bar{\kappa}^4$ and the gauge parameter $\hat{\alpha}$ and scale μ in this case is

$$\ln\frac{\bar{\kappa}^2}{4\pi\mu^2} = -\frac{(4\pi)^2}{\hat{\alpha}\hat{g}^2} + 1 - \gamma_E \tag{6.24}$$

The 1-loop effective actions (6.22) and (6.20) suggest that $-\langle\sigma^a\phi^a\rangle$ and $\langle\bar{\gamma}^a\bar{\gamma}^a\rangle$ play analogous physical roles in the gauges described by the actions $S_G^{(2)}$ and $S_G^{(1)}$. Note that the relations (6.24) and (6.19) determining the position of the maxima of the respective 1-loop effective actions in terms of the scale μ and the gauge parameter $\hat{\alpha}$ differ by $\hat{\alpha} \to 2\hat{\alpha}$. A byproduct of the calculation leading to (6.22) is the renormalization constant of $(\sigma^a\phi^a)$,

$$Z_{\sigma\phi} = 1 - \frac{3\hat{g}^2}{(4\pi)^2\epsilon} + O(\hbar^2) \tag{6.25}$$

Comparing (6.25) with (6.18) shows that the critical exponents of $(\bar{\gamma}^a\bar{\gamma}^a)$ and $\sigma^a\phi^a$ coincide for $\alpha \to 0$. These results are consistent, since the models we are considering should

coincide at the point $\alpha = \beta = 0$ in the gauge-parameter space which corresponds to Landau gauge.

Let us finally investigate effects on the perturbative Green functions of the gauge boson from nonvanishing expectation values $\langle\bar\gamma^a\bar\gamma^a\rangle$ and $\langle\sigma^a\phi^a\rangle$. To this end, we calculate the generalized gluon polarization $\Gamma^{2\,ab}_{\mu\nu}(x-y;\varphi)$ of (6.3) to one loop in the gauge classes defined by $S_G^{(1)}$ respectively $S_G^{(2)}$.

The dependence of the 1-loop gluon polarization on the constant ghost $\bar\gamma$ in the model with action $S_G^{(1)}$ comes from a ghost loop with two gluon vertices and an arbitrary number of insertions of the constant ghost $\bar\gamma$. The only nonvanishing contributions for $SU(2)$ correspond to the Feynman diagrams of Fig. 3.

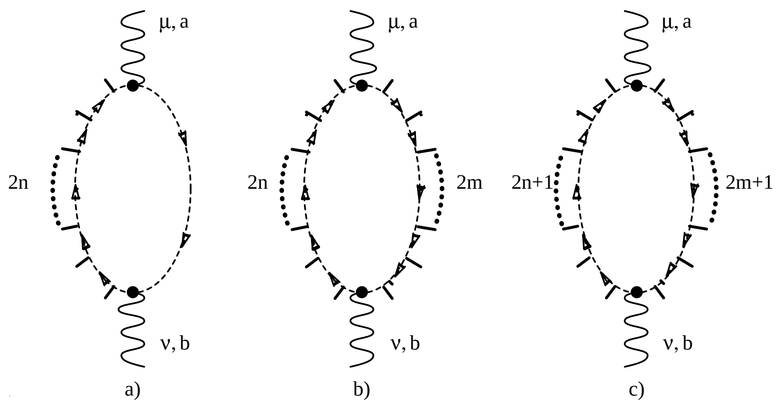

Fig.3: $\bar\gamma$-dependent contributions to the 1-loop gluon polarization for $SU(2)$: a) an even number (≥ 2) of $\bar\gamma$-insertions on one side of the loop only, b) an even number (≥ 2) of insertions on either side of the loop, c) an odd number of insertions on either side. Legend as in Fig. 1.

Summing over the number of insertions for each class of diagrams in Fig. 3 these contributions to the gluon polarization in momentum space correspond to the integral expressions,

$$\begin{aligned}
\Gamma^{2\,ab}_{\mu\nu}(q;a)) &= -2(\bar\gamma^a\bar\gamma^b + \delta^{ab}\kappa^4)\kappa^{-4}\int\frac{d^4k}{(2\pi)^4}\frac{(k+q)_\mu k_\nu}{k^2(k+q)^2}\sum_{n=1}^{\infty}\left(\frac{-\kappa^4}{k^4}\right)^n \\
&= 2(\delta^{ab}\kappa^4 + \bar\gamma^a\bar\gamma^b)\int\frac{d^4k}{(2\pi)^4}\frac{(k+q)_\mu k_\nu}{k^2(k+q)^2(k^4+\kappa^4)} \\
\Gamma^{2\,ab}_{\mu\nu}(q;b)) &= -2\bar\gamma^a\bar\gamma^b\kappa^4\int\frac{d^4k}{(2\pi)^4}\frac{(k+q)_\mu k_\nu}{k^2(k+q)^2(k^4+\kappa^4)((k+q)^4+\kappa^4)} \\
\Gamma^{2\,ab}_{\mu\nu}(q;c)) &= -2\bar\gamma^a\bar\gamma^b\int\frac{d^4k}{(2\pi)^4}\frac{(k+q)_\mu k_\nu}{(k^4+\kappa^4)((k+q)^4+\kappa^4)}
\end{aligned} \tag{6.26}$$

We have replaced $\bar\gamma^a\bar\gamma^a$ in (6.26) by its vacuum expectation value $\kappa^4 = \langle\bar\gamma^a\bar\gamma^a\rangle$ in the thermodynamic limit. Note that the (Euclidean) integrals in (6.26) only make sense for $\kappa^4 > 0$ and are ultraviolet and infrared finite in $D = 4$ dimensions. The fact that these corrections from the constant ghosts are finite proves our previous assertion that the

renormalization constants of the ordinary Yang-Mills theory are unchanged (at least in the minimal scheme and to one loop).

A partial fraction decomposition expresses the moduli-dependent contribution to the gluon polarization,

$$\begin{aligned}\Gamma^{2\,ab}_{\mu\nu}(q) = \quad & \kappa^2 \lim_{D\to 4}\left(\delta_{\mu\nu} - 2q_\mu q_\nu \frac{\partial}{\partial q^2}\right)\Big[\delta^{ab}(2I_D[q^2;0,0] - I_D[q^2;\kappa^2,0] - I_D[q^2;-\kappa^2,0]) \\ & + \frac{\bar{\gamma}^a\bar{\gamma}^b}{\kappa^4}(I_D[q^2;\kappa^2,0] + I_D[q^2;-\kappa^2,0] - 2I_D[q^2;\kappa^2,-\kappa^2])\Big]\end{aligned} \tag{6.27}$$

in terms of the dimensionless scalar integrals

$$I_D[q^2;a,b] = \frac{\Gamma(\frac{2-D}{2})}{8\pi}\int_0^1 dx \left[\frac{x(1-x)q^2 + iax + ib(1-x)}{4\pi\kappa^2}\right]^{\frac{D-2}{2}} \tag{6.28}$$

The limit in (6.27) exists and the leading correction from a nonvanishing value of κ to the asymptotic behaviour of the transverse gluon polarization is

$$\Gamma^{2\,ab}_{\mu\nu}(q^2\to\infty;\bar{\gamma}) \propto \frac{\kappa^4}{q^2}\ln q^2 \tag{6.29}$$

This is not the asymptotic behaviour a simple effective gluon mass would produce. It is remarkable that (6.29) is consistent with the leading power correction to the gluon propagator that arises from a restriction to the Gribov region [1] or the fundamental domain [6].

We have only been considering an unbroken $SU(n)$ gauge group so far and one may wonder whether the constant ghosts could not also lead to noticeable effects in the broken case. Note however that the reasoning leading to the equivariant construction does not really apply to the broken case, because the ω ghost generates constant gauge transformations. An immediate consequence is that the ghosts are generally massive in gauges which depend explicitely on the Higgs field (for example t'Hooft gauges). The constant ghosts $\bar{\gamma}, \sigma$ and ϕ futhermore decouple if the invariance is abelian. These general arguments suggest that one might only expect physical effects from the constant ghosts if an unbroken nonabelian invariance remains. In the following we will only investigate the gauge dependence in an unbroken $SU(2)$ theory by comparing the gluon polarization of models described by $S_G^{(1)}$ and $S_G^{(2)}$ at the one loop level.

For the model described by the classical action $S_G^{(2)}$, the contributions to the gluon polarization from the moduli ϕ and σ are those shown in Fig. 4.

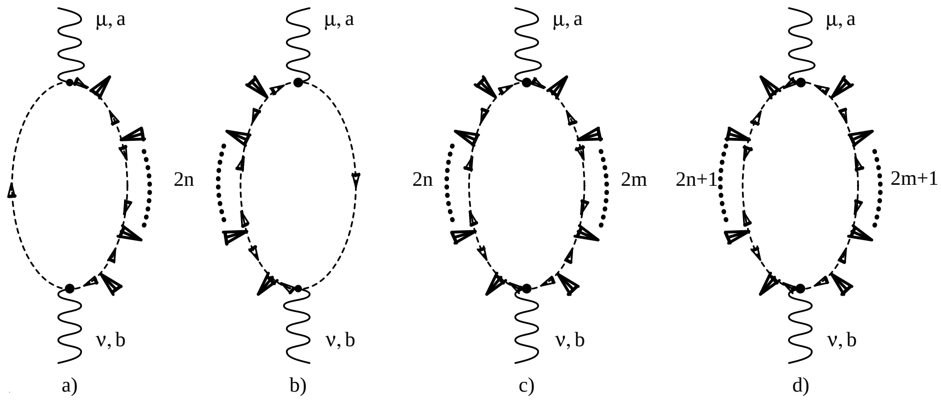

Fig.4: σ and ϕ-dependent contributions to the 1-loop gluon polarization for $SU(2)$: a) an even number (≥ 2) of insertions on one side of the loop only, b) same as a) but with exchanged gluons, c) an even number (≥ 2) of insertions on either side of the loop, d) an odd number of insertions on either side. Legend as in Fig. 2.

The corresponding integral expressions for these polarizations are

$$
\begin{aligned}
\Gamma^{2\,ab}_{\mu\nu}(q;a)) &= (\sigma^a\phi^b - \delta^{ab}\bar{\kappa}^4)\bar{\kappa}^{-4}\int\frac{d^4k}{(2\pi)^4}\frac{(k+q)_\mu k_\nu}{k^2(k+q)^2}\sum_{n=1}^{\infty}\left(\frac{-\bar{\kappa}^4}{k^4}\right)^n \\
&= (\delta^{ab}\bar{\kappa}^4 - \sigma^a\phi^b)\int\frac{d^4k}{(2\pi)^4}\frac{(k+q)_\mu k_\nu}{k^2(k+q)^2(k^4+\bar{\kappa}^4)} \\
\Gamma^{2\,ab}_{\mu\nu}(q;b)) &= (\delta^{ab}\bar{\kappa}^4 - \sigma^b\phi^a)\int\frac{d^4k}{(2\pi)^4}\frac{(k+q)_\mu k_\nu}{k^2(k+q)^2((k+q)^4+\bar{\kappa}^4)} \\
\Gamma^{2\,ab}_{\mu\nu}(q;c)) &= (\delta^{ab}\bar{\kappa}^8\tan^2\theta - \phi^a\phi^b\sigma^2 - \sigma^a\sigma^b\phi^2) \\
&\qquad \int\frac{d^4k}{(2\pi)^4}\frac{(k+q)_\mu k_\nu}{k^2(k+q)^2(k^4+\bar{\kappa}^4)((k+q)^4+\bar{\kappa}^4)} \\
\Gamma^{2\,ab}_{\mu\nu}(q;d)) &= (\sigma^a\phi^b + \sigma^b\phi^a)\int\frac{d^4k}{(2\pi)^4}\frac{(k+q)_\mu(k+q)_\nu}{(k^4+\bar{\kappa}^4)((k+q)^4+\bar{\kappa}^4)}
\end{aligned}
\tag{6.30}
$$

where we have replaced $\sigma^a\phi^a$ by the expectation value $\bar{\kappa}^4 = -\langle\sigma^a\phi^a\rangle > 0$. The dependence of (6.30) on the opening angle θ,

$$
\tan^2\theta = \frac{\vec{\phi}^2\ \vec{\sigma}^2 - (\vec{\phi}\cdot\vec{\sigma})^2}{(\vec{\phi}\cdot\vec{\sigma})^2}
\tag{6.31}
$$

is to be expected in this case, because it is the only other $SU(2)$-invariant with vanishing ghost number that can be formed with $\vec{\sigma}$ and $\vec{\phi}$. The one loop effective action Γ^0 of (6.22) does not depend on this invariant, but (6.30) shows that further loop corrections and thus the full effective action do depend on the angle θ.

Note however that for the special case,

$$
\begin{aligned}
\left\langle\tan^2\theta\right\rangle &= 0 \\
\bar{\kappa}^4 = -\langle\sigma^a\phi^a\rangle &= \langle\bar{\gamma}^a\bar{\gamma}^a\rangle = \kappa^4
\end{aligned}
\tag{6.32}
$$

the two classes of models have *exactly* the same transverse gluon polarization at one loop. To verify this, it is sufficient to observe that the relation (6.32) between the expectation values of the moduli-fields of the two classes of models implies that $\vec{\sigma}$ and $\vec{\phi}$ are anti-parallel and that the length of $\bar{\gamma}$ is the geometric mean of the lengths of σ and ϕ. For such configurations the difference between the gluon polarizations in the two gauges is only in the longitudinal parts of $\Gamma^{2\,ab}_{\mu\nu}(q;c))$ in (6.26) and $\Gamma^{2\,ab}_{\mu\nu}(q;d))$ in (6.30).

The corrections to the transverse gluon polarization affect physical correlation functions such as $\langle F^2(x)F^2(0)\rangle$. We thus believe that (6.32) holds in lowest order, although this can only be verified directly by nonperturbative methods or as the consequence of a Ward identity. The different power corrections to the *longitudinal* gluon polarization are expected due to the intrinsic gauge dependence of this object. They have no physical consequences at this order of perturbation theory. The generalization of these remarks to higher (or all) orders in the loop expansion is clearly desirable but beyond the scope of this paper.

7 Conclusion

We have reinterpreted the Faddeev-Popov procedure as the construction of a TQFT in the gauge group depending on a background connection. To handle the global Faddeev-Popov zero modes in covariant Landau type gauges we considered an equivariant cohomology. The elimination of the constant zero modes is particularly important at finite space-time volume, since they otherwise lead to a vanishing partition function of the Yang-Mills theory. Contrary to pointed gauges [2, 11], our proposal preserves the covariance and translational invariance of gauge dependent Green functions. We believe this to be relevant for defining the infinite volume limit of an unbroken nonabelian gauge theory. It is otherwise not certain that translational invariance of gauge dependent correlators can be recovered in the infinite volume limit, since the color forces of such theories are expected to be strong and of infinite range.

To implement covariant constraints and build the equivariant cohomology we introduced constant ghosts. Most of them can be eliminated from the effective action by equations of motion and the remaining ones may be interpreted as moduli. The topological observables of the gauge group TQFT are elements of descent equations that terminate in operators which are only functions of the constant ghost ϕ which controls the translational symmetry of the Faddeev-Popov ghost. The winding number of gauge group elements is the only topological observable with ghost number zero we can construct in this way (apart from linked observables derived from this one). It is associated with the invariant $\mathrm{Tr}\,\phi^2$ through the descent equations. Using the properties of this observable, we have been able to show in a very simple way that the partition function of the gauge group TQFT (and thus the contribution from the gauge-fixing to the Yang-Mills theory) either vanishes due to generic zero modes or depends on global properties of the connection. The latter situation occurs for space-time manifolds admitting 3-cycles Γ_3 with

$\pi_3(\Gamma) = \mathbf{Z}$. This is precisely the condition found previously by Singer [2] to be associated with a topological Gribov problem.

For an $SU(2)$ gauge group we verified that our improved gauge fixing yields a well-defined and non-vanishing partition function in the vicinity of the trivial Yang-Mills connection. The argument in Appendix A relies on the observation that the partition function of a TQFT calculates the Euler number of the space of fixed points in "delta-" (Landau-) gauge. By analyzing the topological structure of the space of fixed points for $A = 0$ we concluded that its Euler number is *odd*. The partition function of our equivariant TQFT therefore does not vanish generically. Together with (4.9) this gives a direct verification of the dependence of the TQFT on global topological properties in the vicinity of flat background connections. We have shown that this global violation of the BRST symmetry can be reformulated as the existence of a nonlocal but BRST-exact operator whose expectation value does not vanish.

With the modified BRST algebra the usual Feynmann rules are changed by couplings of the constant ghosts to the ordinary Yang-Mills field content. We investigated the possible effects on perturbation theory and gauge invariance by considering two classes of renormalisable gauges involving the constant ghosts. To avoid infrared problems in the perturbative treatment of these gauge theories we reorganized the perturbative expansion and treated constant bosonic ghosts like moduli. Since there is no such argument as a holomorphic dependence on the moduli as in supersymmetric theories, the true effective action for the moduli is not known. However, only a few expectation values of the moduli have to be known in order to completely determine the loop expansion of observables in the infinite volume limit. We computed one-loop effects and observed that the two classes of gauges are physically equivalent provided one suitably identifies expectation values of the constant ghosts. Nonvanishing expectation values of the constant commuting ghosts lead to a modification of the gluon polarization. At sufficiently high momenta, where a perturbative analysis is justified, we obtained power corrections to the leading logarithmic behaviour. These corrections are similar to those one expects nonperturbatively for instance from instantons, but are in our case the result of interactions with the constant bosonic ghosts. Moreover, we verified that the power corrections to the transverse gluon polarization are gauge independent, whereas the modification of the longitudinal polarization is not. This is a nontrivial consistency test for our equivariant construction, but it clearly would be of interest to prove the gauge independence of physical observables beyond the one loop level that we investigated here.

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A The partition function Z[A=0]

We show in this appendix that the partition function

$$\mathcal{Z}[A] = \langle \mathbf{1} \rangle_A = \int [dU][dc][d\bar{c}][db] d\phi d\sigma d\bar{\sigma} d\bar{\gamma} d\gamma \; e^{S_A} \tag{A.1}$$

of the equivariant TQFT with action (3.6) no longer vanishes generically for all connections A and thus establish that (4.9) is non-trivial.

The integrations over the constant ghosts ϕ and σ in (A.1) can be performed, giving a field-independent factor proportional to the inverse volume of space-time (which we take to be finite here). The integration over the other fields is more subtle, since we know that there are (infinitely) many configurations $U(x)$ satisfying the saddle-point equation

$$\partial\cdot A^U = \partial\cdot(U^\dagger A U + U^\dagger \partial U) = 0 \tag{A.2}$$

For $A = 0$ these are the gauge-copies of the vacuum in Landau gauge which were first considered by Gribov [1] who also constructed some of them. The solutions U of (A.2) fall into equivalence classes modulo constant (right) gauge transformations and the equivariant construction is designed to handle this (trivial) degeneracy. A semiclassical evaluation of (A.1) means that one integrates in the vicinity of a representative for each class. This is a difficult task, considering that there are arbitrary many such copies. However, the contribution of a *single* copy to the normalization is expected to be finite due to supersymmetric compensation of the mode expansion. If there were no zero modes, the contribution of *each* copy to (A.1) could be normalized to ± 1, and the result of (A.1) would then be interpreted as the degree of the map of the gauge-fixing [7].

In the more general setting with (additional) zero modes, our equivariant partition function should compute the (generalized) Euler number $\chi(\mathcal{E}_A/SU(n))$ of the topological space of fixed points of (A.2)

$$\mathcal{E}_A := \{U : U(x) \in SU(n), \partial\cdot A^U = 0\} \tag{A.3}$$

rather than the Euler number of $\mathcal{E}_A$ itself [5]. We will argue that $\chi(\mathcal{E}_{A=0}/SU(2)) = odd \neq 0$ for an $SU(2)$ gauge group. The equivariant partition function (A.1) therefore should not vanish in the vicinity[7] of $A = 0$. The action (3.6) at $A = 0$ is also symmetric with respect

[7]It is conceivable that the (generalized) Euler number the partition function computes may depend on topological characteristics of the background connection A. Instructive in this respect is that $\mathcal{E}_A$ can be viewed as the space of fixed points of an associated Morse potential [6] $\mathcal{V}_A[U] \in \mathcal{R}_+$

$$\mathcal{V}_A[U] = \int_{\mathcal{M}} \mathrm{Tr}\, A^U \cdot A^U \tag{A.4}$$

Taken naively, the Morse theorem would indicate that the TQFT is actually calculating the (generalized) Euler number of $\{U : U(x) \in SU(n), V_A[U] < \infty\}/SU(n)$. The Morse potential (A.4) thus resticts the functional space of gauge transformations one is considering in an A-dependent fashion. This statement should however be taken with a grain of salt, since it is far from obvious that Morse theory is applicable in this infinite-dimensional setting.

to *left* multiplication by a (constant) group element $U(x) \to g_L U(x)$. Consequently, if $U(x)$ is a solution to (A.2) at $A = 0$,

$$\partial\cdot(U^\dagger \partial U) = 0 \tag{A.5}$$

then so is

$$U(x, g_L, g_R) = g_L U(x) g_R, \qquad \forall g_L,\, g_R \in SU(n) \tag{A.6}$$

Some of these solutions however belong to the same equivalence class modulo right multiplication by $SU_R(n)$. We obtain the moduli space of these (right equivalence classes of) solutions to (A.5) by noting that left multiplication of $U(x)$ by $g_L \in SU(n)$ is the same as right multiplication by

$$g_R(x) = U^\dagger(x) g_L U(x) \tag{A.7}$$

$g_L U(x)$ therefore belongs to the equivalence class modulo right multiplication of $U(x)$ only if $dg_R(x) = 0$, i.e.

$$[U(x) dU^\dagger(x),\, g_L] = 0 \tag{A.8}$$

Thus left multiplication of $U(x)$ by any $g_L \in SU(n)$ belonging to the subgroup which commutes with $U(x)dU^\dagger(x)$ is redundant. For an $SU(2)$ gauge group there are only three possible subgroups to consider:

1) $g_L \in SU_L(2)$ satisfy (A.8) $\Rightarrow \chi(SU_L(2)/SU_L(2)) = 1$
2) $g_L \in U(1) \subset SU_L(2)$ satisfy (A.8) $\Rightarrow \chi(SU_L(2)/U(1) \simeq S_2) = 2$
3) $g_L \in \{\mathbf{1}, -\mathbf{1}\} \subset SU_L(2)$ satisfy (A.8) $\Rightarrow \chi(SU_L(2)/\{\mathbf{1}, -\mathbf{1}\} \simeq SO(3)) = 0$

Note that case 1) implies that $U(x)dU^\dagger(x) = 0$ and therefore corresponds to the equivalence class of the identity (modulo right multiplication). There is only *one* such class. For an $SU(2)$ gauge group, case 2) can only occur if $U(x)dU^\dagger(x) = d\theta(x)$ is an *abelian* connection. Since $U(x)$ should furthermore satisfy (A.5), we can conclude that this connection vanishes unless there are non-trivial 1-cycles. We thus find that the moduli space of solutions to (A.5) of an $SU(2)$ theory has the topological structure

$$\mathcal{E}_{A=0} = SU_R(2) \times \Big[\mathbf{1} + S_2 \times \mathcal{F} + SO(3) \times \widetilde{\mathcal{F}}\Big] \tag{A.9}$$

In other words $\mathcal{E}_{A=0}/SU(2)$ can essentially be described as a single point and a collection of two- and three-dimensional spheres [2, 3]. Although we do not know the Euler numbers associated with the topological spaces $\mathcal{F}$ and $\widetilde{\mathcal{F}}$, the structure (A.9), together with $\chi(S_2) = 2$ and $\chi(SO(3)) = 0$ suffices to see that

$$\chi(\mathcal{E}_{A=0}/SU(2)) = odd \neq 0 \tag{A.10}$$

In the vicinity of $A = 0$ the degeneracy with respect to left group multiplication is in general lifted, but the signed sum of Morse indices over the (then isolated) fixed points should still give the Euler number (A.10). At least for an $SU(2)$ gauge group, the partition function (A.1) of the TQFT should therefore not vanish near $A = 0$. We thus have explicitly verified (4.9) for the $SU(2)$ case.

References

[1] V.N.Gribov, *Nucl. Phys.* B139 (1978) 1;

[2] I.M.Singer, *Comm. Math. Phys.* 60 (1978) 7;

[3] P. van Baal, *Nucl. Phys.* B369 (1992) 259; Talk given at the ECT workshop on *Non-perturbative Approaches to QCD*, (1995) in Trento, Italy; University of Leiden preprint INLO-PUB-19/95, hep-th/9511119;

[4] K. Fujikawa, *Nucl. Phys.* B223 (1983) 218;

[5] for a review see: D.Birmingham, M.Blau, M.Rakowski and G.Thompson, *Phys. Reports* 209 (1991) 129;

[6] D.Zwanziger, *Nucl. Phys.* B412 (1994) 657; New York University Preprint 1995 NYU-ThPhZ11-95;

[7] K.Fujikawa,*Progr. Theor. Phys.* 61 (1979) 627;
P.Hirschfeld, *Nucl. Phys.* B157 (1979) 37;
B.Sharpe, *J. Math. Phys.* 25 (1984) 3324;

[8] R. Friedberg, T. D. Lee, Y. Pang and H. C. Ren, Columbia report, CU-TP-689; RU-95-3-B;

[9] D. Friedan, *Introduction to Polyakov's String Theory* published in Les Houches Sum.School 1982 *Recent Advances in Field Theory and Statistical Mechanics*;
O. Alvarez,*Nucl. Phys.* B216 (1983) 125;;
A.M. Polyakov, *Gauge Fields and Strings*, Harwood (1987);

[10] L. Baulieu and M. Bellon, *Phys. Lett.* B202 (1988) 67;

[11] P.K. Mitter and C.M. Viallet, *Comm. Math. Phys.* 79 (1981) 457;
M. Daniel and C.M. Viallet, *Rev. Mod. Phys.* 52 (1980) 1;

[12] J.A.Dixon, *Cohomology and Renormalization of gauge theories I, II, III*, Imperial College preprint–1977;
G.Bandelloni, *J. Math. Phys.* 28 (1987) 2775;

[13] S.Coleman and E.Weinberg, *Phys. Rev.* D7 (1973) 1888;

[14] for more general cases see: E. Elizalde and A. Romeo, *Phys. Rev.* D40 (1989) 436;

[15] C.Itzykson and J.–B.Zuber, *Quantum Field Theory*, McGraw–Hill (1980).